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ON REALIZING MODULES OVER THE STEENROD ALGEBRA

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Dedicated to the memory of George Cooke

This paper is a continuation of our study of the decomposition of a co H-space with respect to a self map, begun in [4], where a general theory of such decompositions was developed. If X is a co H-space of finite type (for example the suspension of a space of finite type) $f: X \to X$ a self map, and p a prime or 0 the theory of [4] provides a collection of spaces $X_{p(x)}$, one for each monic irreducible polynomial $p(x) \in \mathbb{Z}/p[x]$, and a map

$$\varphi : X \longrightarrow \bigvee_{p(x)} X_{p(x)}$$

inducing an isomorphism ($\mathbb{Z}/0 \simeq \mathbb{Q}$)

$$\varphi_* : \tilde{H}_*(X; \mathbb{Z}/p) \longrightarrow \bigoplus_{p(x)} \tilde{H}_*(X_{p(x)}; \mathbb{Z}/p),$$

realizing the primary decomposition of $H_*(X; \mathbb{Z}/p)$ with respect to the endomorphism

$$f_* : H_*(X; \mathbb{Z}/p) \to H_*(X; \mathbb{Z}/p),$$

induced by f.

The present portion of our study is concerned with the application of this decomposition to examples arising in the following way. Fix an odd prime p. The $\mathbb{Z}/p$ cohomology of the Eilenberg–Mac Lane complex $K(\mathbb{Z}, 3)$ has been determined by Cartan and Serre [1]. Writing $\iota \in H^3(K(\mathbb{Z}, 3); \mathbb{Z}/p)$ for the mod p fundamental class the cohomology takes the form:

$$H^*(K(\mathbb{Z}, 3); \mathbb{Z}/p) = \mathbb{E}[\iota, P^1\iota, \ldots] \otimes \mathbb{Z}/p[\beta P^1\iota, P^pP^1\iota, \ldots],$$

where P^k is the kth mod p Steenrod operation, β the mod p Bockstein, $\mathbb{E}[\cdots]$ denotes an exterior algebra (on odd dimensional generators) and $\mathbb{Z}/p[\ldots]$ a polynomial algebra. In [3] the construction of spaces realizing successively larger quotients of $H^*(K(\mathbf{Z},3);\mathbf{Z}/p)$ was undertaken. Specifically the following spaces were constructed with the indicated cohomologies:

$$\begin{aligned}
X_0 &= pt; & H^*(X_0;\mathbf{Z}/p) &= \mathbf{Z}/p\\
X_1 &= S^3; & H^*(X_1;\mathbf{Z}/p) &= \mathbf{E}[\iota]\\
X_{2,1} &\ ; & H^*(X_{2,1};\mathbf{Z}/p) &= \mathbf{E}[\iota, P^1\iota]\\
X_{2,p} &\ ; & H^*(X_{2,p};\mathbf{Z}/p) &= \mathbf{E}[\iota, P^1\iota]\otimes\mathbf{Z}/p[\beta P^1\iota]/((\beta P^1\iota)^p)
\end{aligned}$$

(*OBS*: These spaces were denoted by X_i, $i = 0,1,2,3$ in [3].) We will begin by interpolating spaces $X_{2,r}$, $1 < r < p$, between $X_{2,1}$ and $X_{2,p}$ with cohomology

$$H^*(X_{2,r};\mathbf{Z}/p) = \mathbf{E}[\iota, P^1\iota]\otimes\mathbf{Z}/p[\beta P^1\iota]/((\beta P^1\iota)^r),$$

realizing intermediate quotients of $H^*(K(\mathbf{Z},3);\mathbf{Z}/p)$. These spaces are related to each other and to $K(\mathbf{Z},3)$ by maps

$$X_0 \to X_1 \to X_{2,1} \to X_{2,2} \to \cdots \to X_{2,p} \to K(\mathbf{Z},3)$$

inducing the obvious homomorphisms in $\mathbf{Z}/p$ cohomology. Each of these spaces carries a homology $\mathbf{Z}/(p-1)$ action lifted up from multiplication by a primitive root mod p acting on the Eilenberg–Mac Lane space.

We next apply a procedure we call "spreading out the cohomology" (by comparing the degrees of generators in (1.2) and (1.4) it will become clear that the cohomology of E_1 and $E_{2,r}$ has many more gaps than that of X_1 and $X_{2,r}$) by passing to the 3 connective fibrings E_1, $E_{2,r}$ of the spaces X_1, $X_{2,r}$. The homology $\mathbf{Z}/(p-1)$ actions lift to the spaces E_1, $E_{2,r}$, and after suspending these spaces we can apply the decomposition theorem of [4] to obtain eigenspaces of the action. In this way we obtain many spaces realizing small cyclic modules over the Steenrod algebra, such as for example $\mathbf{V}(1)$ (see [12]). By analyzing the cell structure of these spaces we are able to construct a number of interesting relations and elements in the stable homotopy of spheres, Moore spaces and $\mathbf{V}(n)$ spaces.

There are two appendices to the present study. The first is concerned with the following simple problem.

Problem. Let A be an abelian group and X a nice space. Construct in a "natural" way a space $X^{[n]}$ and a map $X^{[n]} \to X$, such that

$$H^i(X^{[n]};A) = \begin{cases} H^i(X;A) & \text{for } 0 \le i \le n,\\ 0 & \text{for } i > n,\end{cases}$$

and the induced map

$$H^i(X;A) \to H^i(X^{[n]};A)$$

is an isomorphism for $i = 0,1,\ldots,n$.

For $A = \mathbf{Z}$ a solution is given by the *homology decomposition* of X (see for example [6]). The case of finite coefficients occurs repeatedly throughout our study. In all cases of interest it was possible to give an ad hoc modification of the homology decomposition that would suffice for our purposes. However these ad hoc arguments were beginning to get out of hand, so we decided to write the first appendix which solves the problem for simply connected X and $A = \mathbf{Z}/p$. We have also worked out a solution using a p-complete version of the homology decomposition on the p-completion of X, however in the interest of simplicity we present a construction in the framework of classical homotopy theory.

The second appendix describes the notation inspired by organic chemistry (atoms and bondings) we adopt for indicating cell structures and Steenrod algebra actions on modules with few elements.

The second author would like to thank the Mathematics Section of the Université de Genève for their kind hospitality during the summers of 1975 and 1976 when portions of this work were done.

Despite the untimely death of George E. Cooke in November of 1976 the second author hopes to bring to a more conclusive point the constructions of spaces realizing small quotients of $H^*(K(\mathbf{Z},3);\mathbf{Z}/p)$ begun by George E. Cooke in [3] and continued in collaboration with him from early 1974 until his death, of which a portion is presented herein.

1. Spaces with interesting cohomology

In this section p will denote a fixed odd prime. We use the notation

$$P^{In} = \begin{cases} P^{p^{n-1}} \cdots P^p P^1 & \text{for } n > 0, \\ 1 & \text{for } n = 0. \end{cases}$$

for the basic iterated Steenrod operation mod p. Note $\deg P^{In} = 2p^n - 2$. Recall that [1]

$$H^*(K(\mathbf{Z},3);\mathbf{Z}/p) = \mathbf{S}[P^{In}\iota, \beta P^{Im}\iota \mid n \geq 0, m > 0],$$

where $\mathbf{S}[\quad]$ is the symmetric algebra functor and ι is the fundamental class, and $\deg P^{In}\iota = 2p^n + 1$, $\deg P^{Im}\iota = 2p^m + 2$. Introduce the two stage Postnikov system

$$Y_r;\ (0 < r < p) \qquad \begin{array}{ccc} Y_r & \longrightarrow & L(\mathbf{Z}, 2r(p+1)-1) \\ \big\downarrow{\scriptstyle q_r} & & \big\downarrow \\ K(\mathbf{Z},3) & \xrightarrow[\varphi_r]{} & K(\mathbf{Z}, 2r(p+1)) \end{array}$$

where (N.B. $\beta P^1\iota$ is the reduction of an integral class)

$$\varphi_r^*(j_{2r(p+1)}) = (\beta P^1\iota)^r$$

and $j_{2r(p+1)} \in H^{2r(p+1)}(K(\mathbf{Z}, 2r(p+1)); \mathbf{Z}/p)$ is the mod p reduction of the fundamental class. (Note that $\deg \beta P^1\iota = \deg \beta P^1 + 3 = 1 + 2(p-1) + 3 = 2(p+1)$.) The system is stable when $r = 1$ but not otherwise.

Theorem 1.1. *In dimensions less than or equal to* $2r(p+1)+2$

$$H^*(Y_r; \mathbb{Z}/p) \cong \mathbf{E}[u, P^1u] \otimes \mathbb{Z}/p[\beta P^1 u]/(\beta P^1 u)^r$$

for $r = 1, \ldots, p-1$; *and hence there is a space* $X_{2,r}$ *(namely a* $2r(p+1)+2$ *p-normal section of* Y_r; *see the Appendix for a discussion of p-normal sections) and a map* $X_{2,r} \overset{q_r}{\hookrightarrow} Y_r$, *such that*

$$H^*(X_{2,r}; \mathbb{Z}/p) \cong E[u, P^1u] \otimes \mathbb{Z}/p[\beta P^1 u]/(\beta P^1 u)^r$$

where $u = q_r^*\iota$ *(where we have written* q_r *for* $q_r \mid X_{2,r}$*).*

Proof. Consider the Eilenberg–Moore spectral sequence [11] (write $H^*(\ )$ for $H^*(\ ;\mathbb{Z}/p)$)

$$E_s \Rightarrow H^*(Y_r)$$

$$E_2 = \operatorname{Tor}_{H^*(K(\mathbb{Z};2r(p+1)))}(H^*(K(\mathbb{Z},3)), \mathbb{Z}/p)$$

for the fibre square $\mathbf{Y}_r$. By computations of Cartan and Serre [1]

$$H^*(K(\mathbb{Z}, 2r(p+1))) \simeq \mathbf{S}[j, P^1_j, \beta P^1_j, \text{etc.}].$$

The map φ_r^* is the tensor product of the map

$$\mathbf{E}[\iota, P^1\iota] \otimes \mathbb{Z}/p[\beta P^1\iota] \leftarrow \mathbb{Z}/p[j]$$

$$(\beta P^1\iota)^r \leftarrow j$$

and a map

$$\mathbf{S}[P^{I_n}\iota, \beta P^{I_n}\iota \mid n \neq 0, 1] \leftarrow \mathbf{S}[P^1_j, \beta P^1 j, \text{etc.}].$$

The E_2 term is therefore the tensor product of

$$A = \operatorname{Tor}_{\mathbb{Z}/p[j]}(\mathbf{S}[\iota, P^1\iota, \beta P^1 i], \mathbb{Z}/p)$$

$$\simeq \mathbf{E}[\iota, P^1\iota] \otimes \mathbb{Z}/p[\beta P^1\iota]/(\beta P^1\iota)^r,$$

using [10; II Section 1] to compute, and

$$B = \operatorname{Tor}_{\mathbf{S}[P^1_j, \text{etc}]}(\mathbf{S}[P^{I_n}\iota, \beta P^{I_n}\iota \mid n \neq 0, 1], \mathbb{Z}/p).$$

Note that

$$\deg P^1 j = 2r(p+1) + 2(p-1),$$

$$\deg P^{I_2}\iota = 2p^2 + 1,$$

and hence for $r < p$ we find (since $p > 2$) that $\operatorname{Tot}(B)$ is connected thru dimension

$$t_r = \min\{2p^2, 2r(p+1)+2p-1\} \geq 2r(p+1)+2,$$

with equality only for $r = p-1$. Degree considerations show there are no differentials thru total degree $2rp(p+1)+2p(p-1)$ and hence

$$H^*(Y_r; \mathbb{Z}/p) = \mathrm{S}[u, P^1u, \beta P^1u]/(\beta P^1u)^r$$

thru dimension t_r and the result follows. □

We turn next to a construction of an analogous space $X_{2,p}$ satisfying

$$H^*(X_{2,p}; \mathbb{Z}/p) \cong \mathrm{S}[u, P^1u, \beta P^1u]/(\beta P^1u)^p.$$

To this end introduce the stable two stage Postnikov system

$$\mathscr{A}\quad \begin{array}{ccc} A & \longrightarrow & L(\mathbb{Z}/p, 2p^2+2p-1) \times L(\mathbb{Z}, 2p^2+2p+1) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}/p, 2p^2+1) & \xrightarrow{\Psi} & K(\mathbb{Z}/p, 2p^2+2p-1) \times K(\mathbb{Z}, 2p^2+2p+1) \end{array}$$

where

$$\Psi^*(x) = P^1 j, \qquad \Psi^*(y) = \beta P^1 \beta j$$

and

$$x \in H^{2p^2+2p+1}(\mathbb{Z}/p, 2p^2+2p-1; \mathbb{Z}/p),$$
$$y \in H^{2p^2+2p+1}(\mathbb{Z}, 2p^2+2p+1; \mathbb{Z}/p),$$
$$j \in H^{2p^2+1}(\mathbb{Z}/p, 2p^2+1; \mathbb{Z}/p),$$

are the fundamental classes, and we have used the fact that $\beta P^1 \beta j$ is the reduction of an integral class. The $\mathbb{Z}/p$ cohomology of A may be computed from the calculations of Cartan and Serre [1] by the method of [10; Section 4, 5] and we find

$$H^*(A; \mathbb{Z}/p) \cong \mathrm{S}[j, \beta j, P^1\beta j] \otimes H,$$

where H is a Hopf algebra that is $2p^2+4p-2$ connected.

Consider the diagram

$$\begin{tikzcd} & A \arrow[d] & \\ K(\mathbb{Z}, 3) \arrow[ur, dashed, "\varphi_p"] \arrow[r, "\varphi_p'"'] & K(\mathbb{Z}, 2p^2+1) \arrow[r, "\Psi"] & \begin{array}{c} K(\mathbb{Z}/p, 2p^2+2p-1) \\ \times \\ K(\mathbb{Z}, 2p^2+2p+1) \end{array} \end{tikzcd},$$

where

$$\varphi_p'^*(j) = P^p P^1 \iota.$$

Recall that

$$P^1P^pP^1\iota = 0,$$

$$P^1\beta P^pP^1\iota = (\beta P^1\iota)^p,$$

so

$$\beta P^1\beta P^pP^1\iota = 0,$$

hence there is a dotted lift. Introduce the fibre square

$$\mathbf{Y}_p \qquad \begin{array}{ccc} Y_p & \longrightarrow & PA \\ \downarrow{\scriptstyle q_p} & & \downarrow \\ K(\mathbb{Z},3) & \xrightarrow[\varphi_p]{} & A \end{array}$$

where $PA \downarrow A$ is the path space fibration. Note

$$\varphi_p^*j = P^pP^1,$$

so

$$\varphi_p^*\beta_j = \beta P^pP^1\iota,$$

$$\varphi_p^*P^1j = P^1\beta P^pP^1\iota = (\beta P^1\iota)^p.$$

Theorem 1.2. *In dimensions less than or equal to* $2p^2+2p+2 = 2p(p+1)+2$

$$H^*(Y_p;\mathbb{Z}/p) \simeq \mathrm{E}[u, P^1u] \otimes \mathbb{Z}/p[\beta P^1u]/(\beta P^1u)^p$$

and hence there is a space $X_{2,p}$ *and a map* $X_{2,p} \hookrightarrow Y_p$ *such that*

$$H^*(X_{2,p};\mathbb{Z}/p) \simeq \mathrm{E}[u, P^1u] \otimes \mathbb{Z}/p[\beta P^1u]/(\beta P^1u)^p,$$

where $u = q_p^*i$.

Proof. Consider the Eilenberg–Moore spectral sequence of the fibre square $\mathbf{Y}_p$ and as in the proof of (4.1) note that

$$E_2 \simeq \begin{array}{c} \mathrm{S}[\iota, P^1\iota, \beta P^1\iota]/(\beta P^1\iota)^p \\ \otimes \\ \mathrm{Tor}_H(\mathrm{S}[P^{1n}\iota, \beta P^{1n}\iota \mid n > 2], \mathbb{Z}/p) \end{array}$$

whence the result follows from degree considerations. □

Remark. The preceding construction of $X_{2,p}$ is different from the one indicated in [3]. This is only a matter of convenience.

Proposition 1.3. *For* $1 \leq r \leq p-1$ *there is a map*

$$j_r : X_{2,r} \to X_{2,r+1}$$

making the diagram

$$\begin{tikzcd} X_{2,r} \arrow[rr, "j_r"] \arrow[dr, "q_r"'] & & X_{2,r+1} \arrow[dl, "q_{r+1}"] \\ & K(\mathbb{Z},3) & \end{tikzcd}$$

commutative.

Proof. Consider first the case $r < p-1$. There is the lifting problem

$$\begin{tikzcd} & X_{2,r+1} \arrow[d, "q_{r+1}"] \\ X_{2,r} \arrow[ur, dashed, "j_r"] \arrow[r, "q_r"] & K(\mathbb{Z},3) \end{tikzcd}$$

and observe from (4.1) and the definitions that

$$q^*_{r+1} : H^*(K(\mathbb{Z},3);\mathbb{Z}) \to H^*(X_{2,r+1};\mathbb{Z})$$

is an isomorphism thru dimensions $2(r+1)(p+1)$. Hence the fibre of q_{r+1} is at least $2(r+1)(p+1)-2$ connected, so all the obstructions lie in zero groups. For $r = p-1$ the fibre has classes in dimensions $2p(p+1)-2$ and $2p(p+1)$ that are potential obstructions, however

$$H^{2p(p+1)-2}(X_{2,p-1};\mathbb{Z}) = H^{2p(p+1)}(X_{2,p-1};\mathbb{Z}) = 0,$$

so as before all obstructions vanish for degree reasons. $\square$

Acknowledgement. The existence of $X_{2,p}$ has also been demonstrated by J. Ewing, J. Harpur and C. Wilkerson, all independently. In fact they show $X_{2,p}$ is an H-space, and Harpur shows [1] it is a factor of the exceptional group F_4 when $p = 3$.

Remark. The Cartan–Serre relation

$$P^1\beta P^p P^1\iota = (\beta P^1\iota)^p,$$

$\iota \in H^3(\mathbb{Z},3;\mathbb{Z}/p)$ the fundamental class, shows it is not possible to construct a space $X_{2,p+1}$ such that

$$H^*(X_{2,p+1};\mathbb{Z}/p) = S[u, P^1u, \beta P^1u]/(\beta P^1u)^{p+1}$$

$\deg u = 3$.

Let $E_{2,r} \downarrow X_{2,r}$ be the 3 connective fibring over $X_{2,r}$, that is $E_{2,r}$ is the fibre of

$$q_r : X_{2,r} \to K(\mathbb{Z},3),$$

$r = 1, \ldots, p$.

Theorem 1.4. *$H^*(E_{2,r};\mathbb{Z}/p) = \mathrm{E}[Z_r, \beta y]\otimes \mathbb{Z}/p[y]$, where*

$$\deg Z_r = 2r(p+1)-1,$$

$$\deg y = 2p^2,$$

$$\deg \beta y = 2p^2+1,$$

and the action of the Steenrod algebra is given by

$$P^1 y = 0, \qquad P^p y = 0, \qquad P^{p^2} y = y^p,$$

$$P^p Z_r = \begin{cases} \beta y; & r = 1,\\ 0; & \text{otherwise,}\end{cases}$$

$$\beta^\varepsilon P^E Z_r = 0; \quad r > 1,$$

$$P^1 \beta y = \begin{cases} Z_p; & r = p,\\ 0; & \text{otherwise,}\end{cases}$$

$$P^p y = 0, \qquad P^{p^2}\beta y = 0,$$

the unstability conditions, and the Cartan formula.

Proof. The map (write $H^*(\ )$ for $H^*(\ ;\mathbb{Z}/p)$)

$$q^*_r : H^*(K(\mathbb{Z},3)) \to H^*(X_{2,r};\mathbb{Z}/p)$$

is epic with kernel and Borel ideal generated by

$$P^{I_n}\iota,\ \beta P^{I_n}\iota; \quad n \neq 0,1$$

$$(\beta P^1\iota)^r$$

whence the usual change of rings argument [9] [10] applied to

$$\mathrm{Tor}_{H^*(K(\mathbb{Z},3))}(\mathbb{Z}/p, H^*(X_{2,r}))$$

gives an isomorphism to

$$\mathrm{Tor}_{S[P^{I_n}\iota,\ \beta P^{I_n}\iota,\ (\beta P^1\iota)^r \mid n\neq 0,1]}(\mathbb{Z}/p,\mathbb{Z}/p)$$

and hence the E_2 term of the Eilenberg–Moore spectral sequence of the fibration

$$E_{2,r} \to X_{2,r} \to K(\mathbb{Z},3)$$

is given by

$$E_2 = \begin{cases} \Gamma(s^{-1}P^{I_n}\iota \mid n \neq 0,1),\\ \qquad\otimes\\ \mathrm{E}[s^{-1}\beta P^{I_n}\iota \mid n \neq 0,1],\\ \qquad\otimes\\ \mathrm{E}[s^{-1}(\beta P^1\iota)^r].\end{cases}$$

The first differential is d^{p-1}, given by

$$d^{p-1}(\gamma_p(s^{-1}P^{I_n}\iota)) = -s^{-1}\beta P^{p^n}P^{I_n}\iota,$$
$$= -s^{-1}\beta P^{I_{n+1}}\iota,$$

and so one computes

$$E_p = \begin{cases} \mathbb{Z}/p[s^{-1}P^{I_n}\iota \mid n \neq 0,1]/(s^{-1}P^{I_n}\iota)^p \\ \otimes \\ \mathrm{E}[s^{-1}\beta P^pP^1\iota, s^{-1}(\beta P^1\iota)^r]. \end{cases}$$

Thus $E_p = E_\infty$ for degree reasons. Set

$$Z_r = s^{-1}(\beta P^1\iota)^r \in H^{2r(p+1)-1}(E_{2,r}; \mathbb{Z}/p)$$
$$y = s^{-1}(P^{I_2}\iota) \in H^{2p^2}(E_{2,r}; \mathbb{Z}/p).$$

The formula

$$\beta(s^{-1}P^{I_2}\iota) = s^{-1}\beta P^{I_2}\iota$$

shows that (N.B. [9, Section 4.4])

$$\beta y = s^{-1}\beta P^{I_2}\iota \in H^{2p^2+1}(E_{2,r}; \mathbb{Z}/p).$$

Note also that

$$P^{p^m}s^{-1}P^{I_m}\iota = s^{-1}P^{I_{m+1}}\iota,$$

so that (N.B. [9, Section 4.4])

$$y^{p^s} = P^{p^s} \cdots P^{p^2}y$$

and hence

$$\mathbb{Z}/p[s^{-1}P^{I_n}\iota \mid n \neq 0,1]/(s^{-1}P^{I_n}\iota)^p \subset E_\infty$$

stacks up to

$$\mathbb{Z}/p[y] \subset H^*(E_{2,r}; \mathbb{Z}/p).$$

The Adem relations

$$P^1(\beta P^pP^1\iota) = (\beta P^1\iota)^p$$
$$P^p\beta P^1\iota = \beta P^pP^1\iota$$

yield the formulas

$$P^1y = Z_r, \qquad r = p,$$
$$P^pZ_r = y, \qquad r = 1.$$

There are no other Steenrod operations; in particular $P^{p^2}\beta y = 0$ which may be seen by setting

$$\Delta = \beta P^1 - P^1\beta$$

and recalling the Adam relation

$$\beta P^n - P^n\beta = \Delta P^{n-1}.$$

Applying this when $n = p^2$ gives

$$\begin{aligned} P^{p^2}\beta y &= \beta P^{p^2} y - \Delta P^{p^2-1} y \\ &= \beta y^p - \Delta(0) = 0 \end{aligned}$$

because P^{p^2-1} belongs to the subalgebra generated by P^1 and P^p so must vanish on y for degree reasons and β is a derivation. □

2. Endomorphisms and an application

We next introduce endomorphisms of the spaces $E_{2,r}$ $r = 1, \ldots, p$ to which the decomposition theorem [4] is applicable. The fractured pieces have interesting cell structures from the viewpoint of stable homotopy theory.

To this end let $\varepsilon \in \mathbb{Z}$ reduce to a generator of $\mathbb{Z}/p^\times$. The commutative diagram (for $r < p$; the case $r = p$ is similar and left to the reader)

$$\begin{tikzcd}
 & \mathbf{Y}_r \arrow[d, "q_r"] & & \\
\mathbf{Y}_r \arrow[ur, dashed, "f"] \arrow[d, "q_r"'] & K(\mathbf{Z},3) \arrow[rr, "\varphi_r"] & & K(\mathbf{Z},2r(p+1)) \\
K(\mathbf{Z},3) \arrow[ur, "\varepsilon"'] \arrow[rr, "\varphi_r"'] & & K(\mathbf{Z},2r(p+1)) \arrow[ur, "\varepsilon"] &
\end{tikzcd}$$

defines the dotted map f which by p-normal approximation (see Appendix 1) defines a map

$$f : X_{2,r} \dashrightarrow X_{2,r},$$

such that

$$\begin{tikzcd}
X_{2,r} \arrow[r, "f"] \arrow[d, "j_r"'] & X_{2,r} \arrow[d, "j_r"] \\
X_{2,r+1} \arrow[r, "f"] & X_{2,r+1}
\end{tikzcd}$$

is commutative. Note that

$$f^*u = \varepsilon u,$$

where

$$H^*(X_{2,r}; \mathbb{Z}/p) = S[u, P^1u, \beta P^1 u]/(\beta P^1 u).$$

From the commutative diagram

$$\begin{tikzcd} X_{2,r} \arrow[r, "f"] \arrow[d, "q_r"] & X_{2,r} \arrow[d, "q_r"] \\ K(\mathbb{Z},3) \arrow[r, "\varepsilon"] & K(\mathbb{Z},3) \end{tikzcd}$$

we obtain upon passing to fibres a map

$$f : E_{2,r} \to E_{2,r},$$

such that

$$f^* : H^*(E_{2,r}; \mathbb{Z}/p) \to H^*(E_{2,r}; \mathbb{Z}/p)$$

is determined by

$$f^*y = \varepsilon y,$$

$$f^*z_r = \varepsilon^r z_r, \quad r = 1, \ldots, p.$$

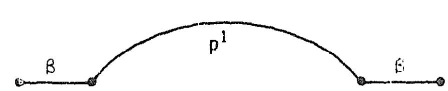
Therefore $(f^*)^{p-1} = 1$ and applying [4, Section 1.2] we obtain a mod p decomposition

$$\Sigma E_{2,r} \simeq \bigvee_{i=1}^{p-1} (E_{2,r})^{\varepsilon^i}$$

into eigenspaces of f.

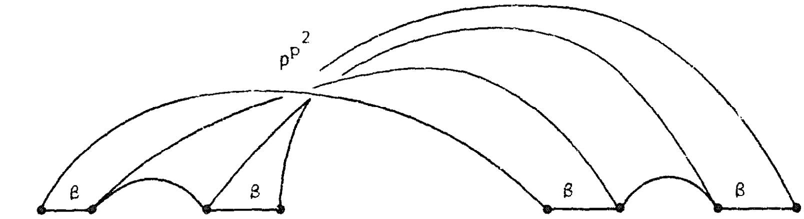
These eigenspaces are interesting from several points of view. We will take this up in earnest in the next section, and we close this section with a single example using $E_{2,p}$.

The cohomology module $H^*(E_{2,p}; \mathbb{Z}/p)_{+1}$ is periodic of period $2p^2(p-1)$, the periodicity being given by multiplication with $y^{p-1} \in H^{2p^2(p-1)}(E_{2,p}; \mathbb{Z}/p)$. The basic block of one period and lowest degree is given by the diagram



class	$y^{p-1}, y^{p-2}\beta y$	$z_p y^{p-2}, z_p y^{p-3}\beta y$
degree	$2p^2(p-1), 2p^2(p-1)+1,$	$2p^2(p-1)+2p-1, 2p^2(p-1)+2p.$

(The vertices of the graph corresponds cohomology classes and the labeled edges to cohomology operations. See appendix II for a more detailed discussion.) Thus a low skeleton of $E^1_{2,p}$ gives a realization of $V(1)$. Using (1.4) one sees that the first two periods are determined by the diagram:



class	y^{p-1}	$z_p y^{p-2}$	$y^{2(p-1)}$	$z_p y^{2p-3}$
degree	$2p^2(p-1)$	$2p^2(p-1)+2p-1$	$4p^2(p-1)$	$4p^2(p-1)+2p-1$

so that we obtain:

Proposition 2.1. *Let p be an odd prime. Then there exists a map (the "periodicity map" for the cell structure of $(\Sigma E_{2;p})^{+1}$)*

$$P^1_{2,p}: S^{2p^2(p-1)-1}\mathbf{V}(1) \to \mathbf{V}(1),$$

such that the functional operation

$$P^{p^2}_{P^1_{2,p}}: \tilde{H}^*(\mathbf{V}(1); \mathbf{Z}/p) \to \tilde{H}^*(S^{2p^2(p-1)-1}\mathbf{V}(1); \mathbf{Z}/p)$$

is an isomorphism. In other words, the mapping cone of $P^1_{2,p}$ has the homotopy type of $\mathbf{V}(2^1_*)/\mathbf{V}(1)$. □

In [15] Toda studies a similar map he calls $\gamma_{[1]}$ for primes $p > 5$ (see in particular p. 235 line 2 and the proposition that follows) defined by the diagram

$$\begin{array}{ccc} S^{2p^3-1}\mathbf{V}(2) & \xrightarrow{\Gamma} & \mathbf{V}(2) \\ \uparrow & & \downarrow \text{collapse} \\ S^{2p^3-1}\mathbf{V}(1) & \xrightarrow[\gamma_{[1]}]{} & S^{2p^2}\mathbf{V}(1) \end{array}$$

where Γ is the basic map whose mapping cone is $\mathbf{V}(3)$. He then goes on to note that $\gamma_{[1]}$ can be defined by special arguments when $p = 5$. Our (1.5) has the advantage that it defines an element

$$P^1_{2,p} \in \pi_{2p^2(p-1)-1}(\mathbf{V}(1), \mathbf{V}(1))$$

for all odd primes without exception. Let $\gamma \in \pi^s_{2p^2(p-1)-2p-1}$ be defined by commutativity of the diagram

$$\begin{array}{ccc} S^{2p^2(p-1)-1}\mathbf{V}(1) & \xrightarrow{P^1_{2,p}} & \mathbf{V}(1) \\ \uparrow & & \downarrow \text{collapse} \\ S^{2p^2(p-1)-1} & \xrightarrow{\gamma} & S^{2p} \end{array}$$

From the results of Zahler and Thomas [17] we then obtain

Proposition 2.2. *The element* $\gamma = \gamma_1 \in \pi^s_{2p^2(p-1)-2p-1}$ *is a non-zero multiple of* $\alpha_1\beta_{p-1}$. *(Note that* $\pi^s_{2p^2(p-1)-2p-1} \cong \mathbf{Z}/p$ *with generator* $\alpha_1\beta_{p-1}$.*)* □

Continuing with our examination of the skeletons of $(\Sigma E_{2,p})^1$ observe that the Z/p cohomology of the first $p-1$ periods is determined by the scheme

$$\boxed{\mathbf{V}(1)} \;\overset{P^{p^2}}{\text{———}}\; \boxed{\mathbf{V}(1)} \;\text{——}\; \cdots \;\overset{P^{p^2}}{\text{——}}\; \boxed{\mathbf{V}(1)}$$

there being $p-1$ copies of $\mathbf{V}(1)$. The Adem relation for $(P^{p^2})^p$ shows $(P^{p^2})^p \in \mathscr{A}^*(P^1, P^p)$ and since both P^1 and P^p vanish on the bottom class it is impossible to attach another $\mathbf{V}(1)$ to the top (or bottom) of this stable complex with a non-zero functional P^{p^2} operation. Thus we have established the following cohomological results and attendant homotopy theoretic consequence.

Proposition 2.3. *For* $0 \leq r \leq p-1$ *the module*

$$\mathbf{E}[Q_0, Q_1] \otimes \mathbf{Z}/p[P^{p^2}]/(P^{p^2})^r$$

over the mod p Steenrod algebra is realizable by a suitable skeleton of $(\Sigma E_{2,p})^1$. □

Proposition 2.4. *The module over the Steenrod algebra mod p*

$$\mathbf{E}[Q_0, Q_1] \otimes \mathbf{Z}/p[P^{p^2}]/(P^{p^2})^p$$

cannot be realized. □

Corollary 2.5. *The Toda bracket*

$$\underbrace{\langle P^1_{2,p}, \ldots, P^1_{2,p}\rangle}_{\longleftarrow\; p \;\longrightarrow} \in \pi^s_{2p^3(p-1)-2}(\mathbf{V}(1), \mathbf{V}(1))$$

is defined and does not contain zero. □

This completes our initial discussion of the eigenspaces $(\Sigma E_{2,p})^1$.

The elements $P^1_{2,p}$ of (2.5) can be used to define a class $\xi \in \pi^s_{(p^3-1)q-4}$, where $q = 2(p-1)$, by the diagram

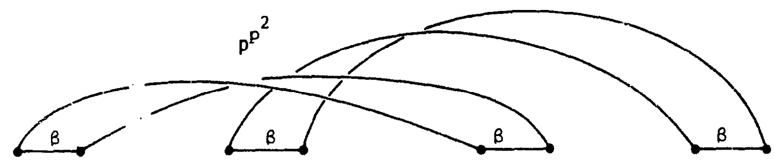
$$\begin{array}{ccc} S^{2p^3(p-1)-2}\mathbf{V}(1) & \xrightarrow{\langle P^1_{2,p}\rangle_p} & \mathbf{V}(1) \\ \uparrow & & \downarrow \\ S^{2p^3(p-1)-2} & \xrightarrow{\quad\xi\quad} & S^{2p} \end{array}$$

It would be interesting to know if this class is non-zero.

3. The spaces $(\Sigma E_{2,r})^1 : 1 < r < p-1$; applications

The Z/p cohomology of the spaces $E_{2,r}$ was computed in (1.4) and in Section 2 an endomorphism $f : E_{2,r} \to E_{2,r}$ was constructed, such that $(f^*)^{p-1} = id$. The $+1$ eigenspace of $H^*(E_{2,r}; \mathbf{Z}/p)$ is periodic with period $2p^2(p-1)$; the periodicity is given by multiplication with $y^{p-1} \in H^{2p^2(p-1)}(E_{2,r}; \mathbf{Z}/p)$. The first two periods are determined by the diagram

Diagram I:

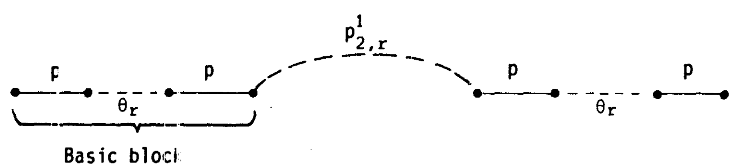


class: $z_r y^{p-r-1}$ $\quad y^{p-1}$ $\quad y^{p-1}\beta y$ $\quad z_r y^{2p-r-2}$ $\quad y^{2p-2}$

deg: $q(p^2-rp)+2r-1$ $\quad p^2 q$ $\quad q(2p^2-rp)+2r-1$ $\quad 2p^2 q$

where $q = 2(p-1)$. The corresponding cell structure for $(\Sigma E_{2,r})^1$ looks stabely as follows:

Diagram II:



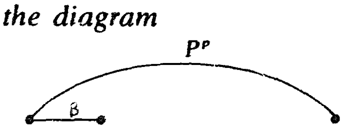
stable dim 0 $\quad rpq-2r+1$ $\quad qp^2$ $\quad qp^2+rqp-2r+1$

The map

$$\theta_r \in \pi_{rqp-2r}(\mathbf{V}(0), \mathbf{V}(0))$$

is essential for the understanding of one "block" of the cell structure (the "basic block"), and the map $P^1_{2,r}$ for understanding how these basic blocks are put together. The following fundamental fact will be the basis of our discussion of these maps.

Theorem of Toda [14]. *There is no complex whose mod p Steenrod algebra action is given by the diagram*



(The "Forbidden Complex")

Lemma 3.1. *In the notations preceding, the attaching map θ_r must be non-zero in* $\pi_{r(qp-2)}(\mathbf{V}(0), \mathbf{V}(0))$.

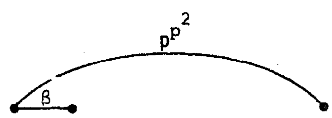
Proof. If $\theta_r \sim 0$, then stably

$$P^1_{2,r}: S^{qp^2-1} \longrightarrow \mathbf{V}(0) \vee \Sigma^{rqp-2r+1}\mathbf{V}(0)$$

and the mapping cone of the composition

$$S^{qp^2-1} \longrightarrow \mathbf{V}(0) \vee \Sigma^{rqp-2r+1}\mathbf{V}(0) \to \mathbf{V}(0)$$

gives a complex with cohomology structure



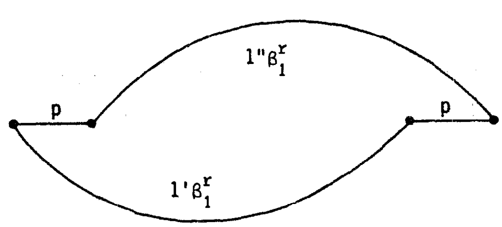
which is a contradiction. □

Consulting a good set of Japanese tables (such as [13] [8] [16]) allows θ_r to be determined in several different ways.

Lemma 3.2. *Let $B^1_{2,r} = \mathbf{V}(0) \cup_{\theta_r} C\, S^{rpq-2r}\mathbf{V}(0)$ be the stable basic block of $(\Sigma E_{2,r})^{+1}$. Then the cell structure of $B^1_{2,r}$ is given by the diagram*

$1' \not\equiv 0(p)$ except perhaps on the bottom block

$1'' \not\equiv 0(p)$ except perhaps on the pth block; ie on all but the top block of the first group of blocks



stable dim: 0 $\qquad$ $rpq - 2r + 1 = r(pq - 2) + 1$

Proof. We use the fact

$$(*) \qquad \begin{gathered} \pi^s_{rpq-2r-1} = 0 = \pi^s_{rpq-2r+1} \\ \pi_{rpq-2r} \cong Z/p \text{ with generator } \beta_1^r. \end{gathered}$$

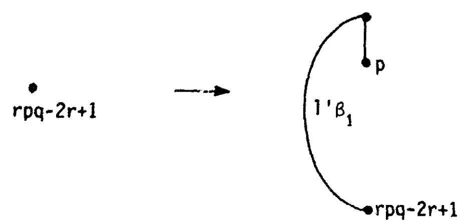
The $rpq - 2r + 1$ cell is attached to the bottom $V(0)$ by

$$S^{rpq-2r} \longrightarrow S^0 \cup_p e^1.$$

By (*) it attaches only to S^0 by a multiple of β_1^r. The $r(pq-2)+2$ cell is then attached by a map

$$S^{rpq-2r+1} \longrightarrow e^{rpq-2r+1} \cup_{1'\beta_1^r} S^0 \cup_p e^1$$

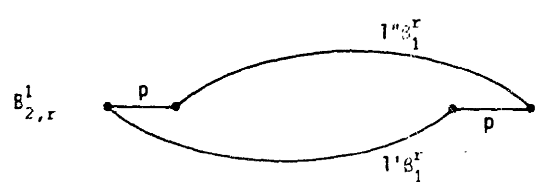
or diagrammatically



Squeeze the bottom cell to a point to get

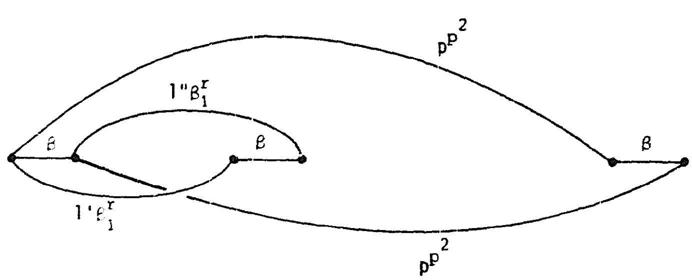
$$S^{rpq-2r+1} \to S^{rpq-2r+1} \vee S^1.$$

The cohomology structure gives the attaching map on the top cell as p; condition (*) on the 1 cell as $1''\beta_1^r$. Finally by (*) and a cofibration argument we see $S^{rpq-2r+1}$ does not attach to the 0 cell. Thus

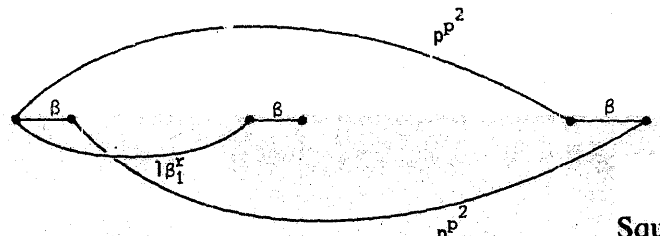


and it remains to show $1'1'' \not\equiv 0$ mod p.

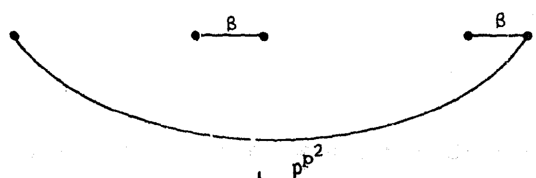
To this end note that the low dimensional stable cell cohomology structure of $(\Sigma E_{2,r})^{-1}$ is partially determined and corresponds to the diagram



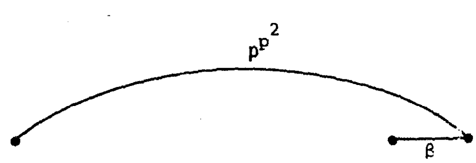
$1'' \not\equiv 0 \bmod p$: Suppose to the contrary $1'' \equiv 0 \bmod p$. So the complex we have constructed is (on all but the top block)



Squeeze the bottom cell to a point giving the map and complex.



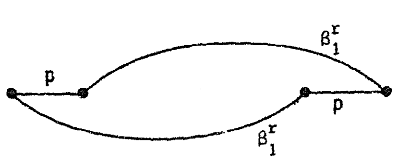
The middle Moore space does not attach to the new bottom cell, so we may squeeze the middle Moore space to a point giving the Spanier Whitehead dual of the forbidden complex



which is a contradiction.

$1' = \text{mod } p$: On all but the bottom block we can apply the Spanier–Whitehead dual of the preceding to the given block and the block to its left. □

Remark. Esthetic considerations suggest that in fact $1'$ and $1'' \not\equiv 0 \bmod p$ on all the blocks, so they are all homotopy equivalents to



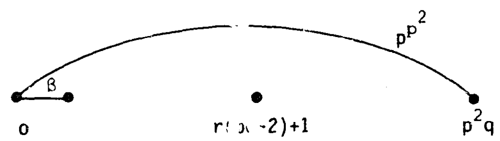
but we have been unable to prove this as yet.

Lemma 3.3. *The composite*

$$S^{p^2q-1} \xrightarrow[P^1_{2,r}]{} B^1_{2,r} \xrightarrow[c]{} S^{r(pq-2)+2}$$

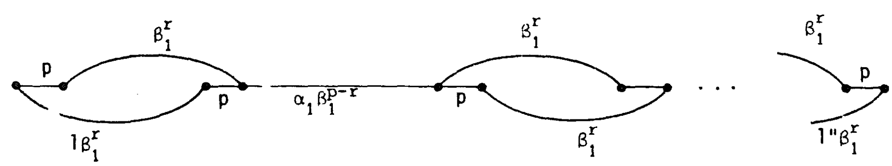
is a nonzero multiple of $\alpha_1\beta_1^{p-r}$.

Proof. The composite in question lies in [13] $\pi^s_{q(p^2-rp)+2r-3} \cong Z/p$ with generator $\alpha_1\beta_1^{p-r}$. Assume the composite is zero. That means $P^1_{2,r}$ pulls off the top cell of $B^1_{2,r}$ and we can construct the complex



According to [13] $\pi^s_{q(p^2-rp)+2r-2}=0$ so $P^1_{2,r}$ compresses to the bottom V(0) where its mapping cone becomes the forbidden complex. ☐

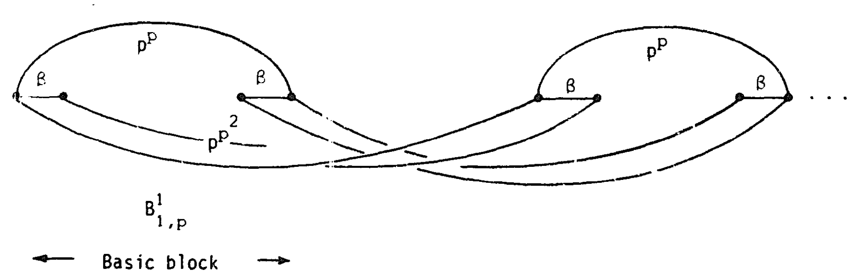
Proposition 3.4. *The low dimensional cell structure of* $(\Sigma E_{2,r})^1$ *is stabely given by*



Therefore the complex $B^1_{2,r} \cup_{P^1_{2,r}c} \Sigma^{p^2q-1} B^1_{2,r}$ gives a realization of Toda's important relation $\alpha_1\beta_1^p=0$. ☐

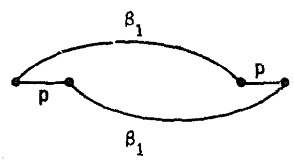
4. The space $(\Sigma E_{1p})^1$; applications

The cohomology of $(\Sigma E_{1,p})^1$ is determined by (1.4) and corresponds to the diagram



The structure of the basic block is determined as an easy consequence of Toda's theorem.

Lemma 4.1. *The stable cell structure of $B^1_{1,p}$ is given by the diagram*



$\square$

The low cohomology structure gives:

Proposition 4.2. *There is a map*

$$\theta_1 \in \pi_{pq-2}(\mathrm{V}(0), \mathrm{V}(0)),$$

with non-zero functional P^p operation. $\square$

The mapping cone of θ_1 is the basic block $B^1_{1,p}$ and looking further at the cohomology gives.

Proposition 4.3. *There is a map*

$$P^1_{1,p} \in \pi_{p^2q-1}(B^1_{1,p}, B^1_{1,p}),$$

such that the functional operation

$$P^{p^2}_{P^1_{1,p}} : \tilde{H}^*(B^1_{1,p}; \mathbf{Z}/p) \to H^*(\Sigma^{p^2q} B^1_{1,p}; \mathbf{Z}/p)$$

is an isomorphism. $\square$

Consider the element $\xi \in \pi^s_{q(p^2-p)-1}$ defined by the diagram

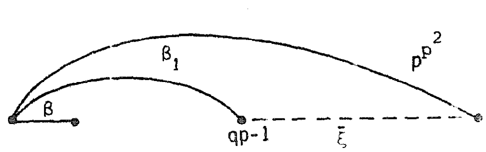
$$\begin{tikzcd} \Sigma^{p^2q-1} B^1_{1,p} \arrow[r, "P^1_{1,p}"] & B^1_{1,p} \arrow[d] \\ S^{p^2q-1} \arrow[u, hook] \arrow[r, "\xi"] & S^{pq} \end{tikzcd}$$

Proposition 4.4. *In the notations preceding, $\xi \in \pi^s_{q(p^2-p)-1}$ is a non-zero multiple of $\gamma_1 = \alpha_1\beta_{p-1}$.*

Proof. According to [13] $\pi^s_{q(p^2-p)-1} \simeq \mathbf{Z}/p$ with generator $\gamma_1 = \alpha_1\beta_{p-1}$. Let $\mathrm{W}(1) = B^1_{1,p}$ and $\mathrm{W}(\frac{1}{2})$ be $\mathrm{W}(1)$ minus the top cell. If $\xi = 0$, then

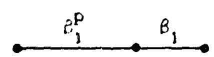
$$P^1_{1,p}\iota : S^{p^2q-1} \longrightarrow \mathrm{W}(\tfrac{1}{2})$$

and the mapping cone is the complex



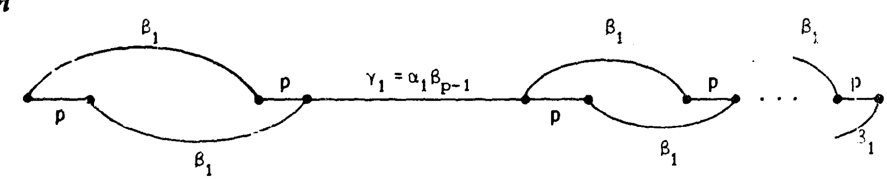
Then $\bar{\xi} \in \pi^s_{q(p^2-p)} \simeq \mathbf{Z}/p$ generated by β_1^p must be essential; otherwise in the preceding complex the $qp-1$ cell can be squeezed out giving the forbidden complex. The Spanier Whitehead dual of this complex is

Note that as [8] $\pi^s_{p^2q-2} = 0$ the p^2q-1 cell attaches trivially to the 0 cell. By duality it is attached trivially to the $q(p-1)$ cell, so we can squeeze it out to get the complex



showing $\beta_1^{p+1} = 0$ contrary to the calculations of Oka [8, Section 6.2]. Therefore the original supposition that [illegible] $= 0$ must be incorrect. $\square$

Corollary 4.5. *The low dimensional stable cell structure of* $(\Sigma E_{2,1})^{+1}$ *is given by the diagram*



This has a number of consequences, among which we mention the following matrix bracket results:

$$0 \in \left\langle (p, \beta_1), \begin{pmatrix} \beta_1 \\ p \end{pmatrix}, \gamma_1 \right\rangle$$

$$\vdots$$

$$0 \in \left\langle (p, \beta_1), \begin{pmatrix} \beta_1 \\ p \end{pmatrix}, \gamma_1, (p, \beta_1), \begin{pmatrix} \beta_1 \\ p \end{pmatrix}, \ldots, \gamma_1 \right\rangle_{p-1}$$

$$0 \notin \left\langle (p, \beta_1), \begin{pmatrix} \beta_1 \\ p \end{pmatrix}, \gamma_1, \ldots, \gamma_1 \right\rangle_p$$

where the subscript indicates the number of times the basic block of 3 terms

$$(p, \beta_1), \begin{pmatrix} \beta_1 \\ p \end{pmatrix}, \gamma_1$$

is repeated in the bracket.

Appendix 1

Our objective in this appendix is to redo the theory of normal decompositions of cw complexes, due to Hilton [6, Section 8], with integral homology replaced by mod p cohomology.

Definition n. Let K be a 1-connected cw complex of finite type and p a prime. A *normal decomposition mod p* of K is a filtration by 1-connected subcomplexes

$$K_2 \subset K_3 \subset \cdots \subset K_n \subset \cdots \subset K$$

with union K such that

$$H^r(K_n; \mathbb{Z}/p) = 0; \quad r > n$$
$$i^* : H^r(K; \mathbb{Z}/p) \xrightarrow{\cong} H^r(K_n; \mathbb{Z}/p); \quad r \leq n$$

where $i : K_n \to K$ is the inclusion. If K admits a normal decomposition mod p we say that K is *normal mod p*, or p-normal, and K_n, is the nth *p-normal section of K*.

Roughly speaking we wish to show that every mod p homotopy type contains a p-normal complex. More precisely we will prove:

Proposition A1. *Let K be a simply connected* cw *complex of finite type p a prime and n a positive integer. Then there exists a simply connected p-local* cw *complex* $K^{[n]}$ *of finite type and a map*

$$i_{[n]} : K^{[n]} \to K,$$

such that

$$H^i(K^{[n]}; \mathbb{Z}/p) = 0; \quad i > n,$$
$$i^*_{[n]} : H^i(K; \mathbb{Z}/p) \xrightarrow{\cong} H^i(K^{[n]}; \mathbb{Z}/p); \quad i \leq n.$$

Moreover this construction is natural in the following sense. Given a map $g : K' \to K''$ there exists a diagram

$$\begin{tikzcd} K'^{[n]} \arrow[r, "g^{[n]}"] \arrow[d, "i'_{[n]}"'] & K''^{[n]} \arrow[d, "i''_{[n]}"] \\ K' \arrow[r, "g"'] & K'' \end{tikzcd}$$

which commutes upon passing to $\mathbb{Z}/p$ cohomology.

N.B. Suppose that

$$K_2 \subset \cdots \subset K_n \subset \cdots \subset K$$

is a normal decomposition of K in the sense of Hilton for $\mathbb{Z}$ homology. Examining the diagram

$$\begin{tikzcd} & & & 0 \arrow[d, equal] & \\ 0 \arrow[r] & H_{n+1}(K_n; \mathbb{Z}) \otimes \mathbb{Z}/p \arrow[r] \arrow[d] & H_{n+1}(K_n; \mathbb{Z}/p) \arrow[r] \arrow[d] & H_n(K_n; \mathbb{Z}) * \mathbb{Z}/p \arrow[r] \arrow[d, "\approx"] & 0 \\ 0 \arrow[r] & H_{n+1}(K; \mathbb{Z}) \otimes \mathbb{Z}/p \arrow[r] & H_{n+1}(K; \mathbb{Z}/p) \arrow[r] & H_n(K; \mathbb{Z}) * \mathbb{Z}/p \arrow[r] & 0 \end{tikzcd}$$

readily shows two things

$$1^0: H_{n+1}(K_n;\mathbb{Z}/p) \text{ can be non-zero}$$

$$2^0: H_{n+1}(K_n;\mathbb{Z}/p)\to H_{n+1}(K;\mathbb{Z}/p) \text{ can fail to be iso.}$$

Therefore K_n need be neither an nth normal p-section nor an $n+1$st normal p section of K.

Recall that a map $f: X\to Y$ is called a *p equivalence* if $f^*: H^*(Y;\mathbb{Z}/p)\to H^*(X;\mathbb{Z}/p)$ is an isomorphism. (N.B. p equivalence is not in general an equivalence relation; see [7].)

Corollary A.2. *Let L be a simply connected* cw *complex of finite type. Then there exists a p normal complex K and a p equivalence $f: K\to L$. Moreover, if $g: L'\to L''$ is a map between simply connected* cw *complexes of finite type, then there exists a commutative diagram*

$$\begin{array}{ccc} K' & \xrightarrow{G} & K'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f''} \\ L' & \xrightarrow{g} & L'' \end{array}$$

where K', K'' are p-normal, f', f'' are p-equivalences, and $G(K'_n)\subset K''_n$. □

We will require several preliminary constructions before taking up the proof of (A1). To begin let

$$\rho_t: H^*(\ ;\mathbb{Z}/p^t)\to H^*(\ ;\mathbb{Z}/p)$$

be the reduction map for $t=1,2,\ldots$ and let

$$\rho: H^*(\ ;\mathbb{Z})\to H^*(\ ;\mathbb{Z}/p)$$

be the integral reduction map. For a cw complex K of finite type define a decreasing filtration

$$F^tH^*(K;\mathbb{Z}/p)=Im\{\rho_t: H^*(K;\mathbb{Z}/p^t)\to H^*(K;\mathbb{Z}/p)\}.$$

It is important to note that as K is of finite type

$$\bigcap F^tH^*(K;\mathbb{Z}/p)=Im\{\rho: H^*(K;\mathbb{Z})\to H^*(K;\mathbb{Z}/p)\}$$

and that the filtration is stable in the sense that for each m there exists a $t(m)$, such that

$$F^tH^m(K;\mathbb{Z}/p)=F^\infty H^*(K;\mathbb{Z}/p);\quad t>t(m)$$

where

$$F^\infty H^*(K;\mathbb{Z}/p)=Im\{H^*(K;\mathbb{Z})\to H^*(K;\mathbb{Z}/p)\}.$$

Thus there is an isomorphism

$$H^m(K;\mathbb{Z}/p) \simeq \bigoplus_{t=0}^{t(m)} E_0^t H^m(K;\mathbb{Z}/p) \oplus F^\infty H^m(K;\mathbb{Z}/p).$$

Let

$$d_t(m) = \begin{cases} \dim_{\mathbb{Z}/p} E_0^t H^m(K;\mathbb{Z}/p) & 1 \leq t \leq t(m) \\ \dim_{\mathbb{Z}/p} F^\infty H^m(K;\mathbb{Z}/p) & t = t(m) \end{cases}$$

and

$$A_m = \begin{cases} \mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p & d_1(m) \text{ factors} \\ \qquad \oplus & \\ \qquad \vdots & \\ \mathbb{Z}/p^s \oplus \cdots \oplus \mathbb{Z}/p^s & d_s(m) \text{ factors} \\ \qquad \oplus & \\ \qquad \vdots & \\ \mathbb{Z}/p^{t(m)} \oplus \cdots \oplus \mathbb{Z}/p^{t(m)} & d_{t(m)}(m) \text{ factors} \\ \qquad \oplus & \\ \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & d_{t(m)+1}(m) \text{ factors} \end{cases}$$

Lemma A.3. *There is a map*

$$f_m : K \to K(A_m, m),$$

such that

$$f_m^* : H^i(K(A_m, m);\mathbb{Z}/p) \to H^i(K;\mathbb{Z}/p)$$

is an isomorphism for $i = m$ and a monomorphism for $i = m+1$.

Proof. Choose a basis

$$\{u_{r,s} \mid r = 1, \ldots, d(s);\ s = 1, \ldots, t(m)+1\}$$

for $H^m(K;\mathbb{Z}/p)$ such that $u_{r,s} \in F^s$, $r = 1, \ldots, d(s)$ and their equivalence classes in F^s/F^{s-1} are a basis $s = 1, \ldots, t(m)$, and $u_{1,t(m)+1}, \ldots, u_{d(t(m)+1),t(m)+1}$ is a basis for $F^{t(m)+1} = F^\infty$. Choose classes

$$v_{r,s} \in \begin{cases} H^m(K;\mathbb{Z}/p^s) & 1 \leq s \leq t(m), \\ H^m(K;\mathbb{Z}) & s = t(m)+1, \end{cases}$$

reducing to $u_{r,s}$ and let

$$f_m^{r,s} : K \to \begin{cases} K(\mathbb{Z}/p^s; m) & 1 \leq s \leq t(m), \\ K(\mathbb{Z}; m) & s = t(m)+1, \end{cases}$$

represent $v_{r,s}$. The map

$$f_m : K \to K(A_m ; m),$$

whose components are $f_m^{r,s}$, $r = 1, \dots, d(s)$, $s = 1, \dots, t(m)+1$, is then the requisite map. □

Before proceeding further let us examine the limited naturality of this construction. Suppose $g : K' \to K''$ is a map of simply connected cw complexes of finite type. We propose to construct the dotted map

$$\begin{tikzcd} K' \arrow[r, "g"] \arrow[d, "f'_m"'] & K'' \arrow[d, "f''_m"] \\ K(A'_m, m) \arrow[r, dashed, "G"] & K(A''_m, m) \end{tikzcd}$$

such that upon passing to $\mathbb{Z}/p$ cohomology the diagram commutes. To this end observe (with the obvious prime, double prime notations) classes corresponding to u''_{rs} and

$$g^* u''_{r,s} = \sum_{\substack{\sigma \geq s \\ 1 \leq r \leq d(\sigma)}} \lambda_{\rho,\sigma} u'_{\rho,\sigma}$$

where $\lambda_{\rho,\sigma} \in \mathbb{Z}$. Now let

$$i''_{r,s} \in \begin{cases} H^m(A''_m, m ; \mathbb{Z}/p^s) & 1 \leq s \leq t''(m), \\ H^m(A''_m, m ; \mathbb{Z}) & s = t''(m)+1, \end{cases}$$

$$i'_{\rho,\sigma} \in \begin{cases} H^m(A'_m, m ; \mathbb{Z}/p^s) & 1 \leq s \leq t'(m), \\ H^m(A'_m, m ; \mathbb{Z}) & s = t'(m)+1, \end{cases}$$

be the fundamental $u'_{\rho,\sigma}$. Note that as $\sigma \geq s$, the classes $i'_{\rho,\sigma}$ have well defined $\mathbb{Z}/p^s$ reductions, and hence the class

$$\sum_{\substack{\sigma \geq s \\ 1 \leq \rho \leq d(\sigma)}} \lambda_{\rho,\sigma} i'_{\rho,\sigma} \in H^m(A'_m, m ; \mathbb{Z}/p^s)$$

is well defined, modulo abuse of notations. The desired map G can now be defined by the requirement

$$G^* i''_{r,s} = \sum_{\substack{\sigma \geq s \\ 1 \leq \rho \leq d(\sigma)}} \lambda_{\rho,\sigma} i'_{\rho,\sigma} \in H^m(A'_m, m ; \mathbb{Z}/p^s); \quad 1 \leq s \leq t''(m)+1$$

$$= \sum_{1 \leq \rho \leq d(t'(m)+1)} \lambda_{\rho,t'(m)+1}\, i'_{\rho,t'(m)+1} \in H^m(A'_m, m ; \mathbb{Z}); \; s = t''(m)+1.$$

Thus we have proven:

Lemma A.4. *If K', K'' are simply connected* CW *complexes of finite type, p a prime, m a positive integer and $g : K' \to K''$ a map, then there exists a diagram*

$$\begin{tikzcd} K' \arrow[r, "g"] \arrow[d, "f'_m"'] & K'' \arrow[d, "f''_m"] \\ K(A'_m, m) \arrow[r, "G"] & K(A''_m, m) \end{tikzcd}$$

which commutes upon passing to $\mathbb{Z}/p$ cohomology. $\square$

N.B. From (A.3) it follows that the map

$$G^* : H^*(A''_m, m; \mathbb{Z}/p) \to H^*(A'_m; m; \mathbb{Z}/p)$$

given by (A.3) is unique.

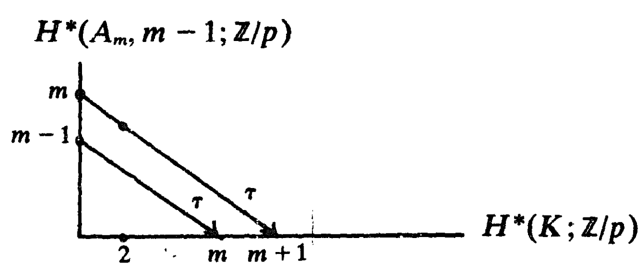
Introduce the fibre square

$$\begin{tikzcd} K\langle m\rangle \arrow[r] \arrow[d, "\pi_m"'] & L(A_m, m-1) \arrow[d] \\ K \arrow[r, "f_m"'] & K(A_m, m) \end{tikzcd}$$

defining the space $K\langle m\rangle$ and the fibration

$$K(A_m; m-1) \to K\langle m\rangle \xrightarrow{\pi_m} K.$$

The E_2 term of the mod p Serre spectral sequence of this fibration looks as follows:



where τ is the transgression. By construction

$$\tau : H^{m-1}(A_m, m-1; \mathbb{Z}/p) \xrightarrow{\sim} H^m(K; \mathbb{Z}/p)$$

$$\tau : H^m(A_m, m-1; \mathbb{Z}/p) \hookrightarrow H^{m+1}(K; \mathbb{Z}/p)$$

and hence we find:

Lemma A.5. *With the notations preceding*

$$\pi_n^* : H^i(K; \mathbb{Z}/p) \to H^i(K\langle m\rangle; \mathbb{Z}/p)$$

is an isomorphism for $i < m$, and $H^m(K\langle m\rangle; \mathbb{Z}/p) = 0$. $\square$

Proof of (A.1). Define inductively

$$\begin{tikzcd} K\langle n+1,\dots,n+m\rangle = K\langle n+1,\dots,n+m-1\rangle\langle n+m\rangle \arrow[dr] & & \arrow[dl] \\ & K\langle n+1,\dots,n+m-1\rangle. & \end{tikzcd}$$

There is then a convergent [20; 8.4] tower of fibrations

$$\begin{array}{c} \vdots \\ K\langle n+1,\dots,n+m\rangle \\ \downarrow \\ \vdots \\ K\langle n+1\rangle \\ \downarrow \\ K \end{array}$$

and letting $K^{[n]} = K\langle n+1,\dots,\infty\rangle$ with $i_{[n]} : K^{[n]} \to K$ the natural map, the result is immediate. □

Using the technique of localization a more elementary proof, in the spirit of [6, Section 8], of a localized version of (A.1) can be given. Specifically,

Proposition A.6. *Let K be a simply connected* CW *complex of finite type, p a prime, and n a positive integer. Then there exists a simply connected* CW *complex* $K_p^{[n]}$ *and a map*

$$i^p_{[n]} : K_p^{[n]} \to K_{(p)},$$

where $K_{(p)}$ *is the localization of K at p, such that*

$$H_i(K_p^{[n]}; \mathbb{Z}/p) = 0; \quad i > n$$

$$i^p_{[n]*} : H_i(K_p^{[n]}; \mathbb{Z}/p) \xrightarrow{\cong} H_i(K_{(p)}; \mathbb{Z}/p)$$

and moreover the construction is natural up to p local equivalence.

Proof. Write

$$H_i(K_{(p)}; \mathbb{Z}) \simeq F_i \oplus T_i; \quad i \geq 0$$

where F_i is a free Q_p module and T_i a finite abelian p group. Choose F''_i, F'_i be free Q_p modules and a map $d_i : F''_i \to F'_i$ such that

$$0 \to F''_i \xrightarrow{d_i} F'_i \to T_i \to 0$$

an exact and minimal. Define a chain complex $\{C_*; \partial_*\}$ by

$$C_i = F_i \oplus F'_i \oplus F''_{i-1}$$

and the requirement

$$\begin{array}{ccccc} C_i = F_i & \oplus & F'_i & \oplus & F''_{i-1}(x, y, z) \\ \downarrow{\scriptstyle \partial_i} & & \downarrow & & \downarrow \\ C_{i-1} = F_{i-1} & \oplus & F'_{i-1} & \oplus & F''_{i-2}(0, d_i z, 0). \end{array}$$

Using p local cells we can make a p local cw complex $\bar{K}_{(p)}$ such that its cellular chain complex is $\{C_*; \partial_*\}$, and there is a map

$$\bar{K}_{(p)} \to K_{(p)}$$

inducing an isomorphism of Q_p homology. Moreover

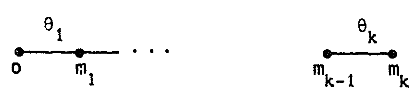
$$\partial_* \otimes \mathbb{Z}/p = 0$$

by minimality, so $H_*(K_{(p)}; \mathbb{Z}/p) \simeq C_*$, and therefore the n-skeleton $\bar{K}^n_{(p)}$ of $\bar{K}_p$ is the required space. □

Appendix 2: Diagrams for cell structures

In discussing cell structures it is convenient to introduce schematic representations for attaching data. This has already appeared to some extent in the preceding section. In an attempt to maximize the utility of such schematic representations, and minimize technical encumberances we will forgo an explicit definition in the general case. The following examples should suffice to allow a rigorous reconstruction of our use of such diagrams here.

Example 1. Suppose given elements $\theta_i \in \pi^s_{n_i}$, $1 \leq i \leq k$. We say that a stable complex X has type $\theta_1, \ldots, \theta_k$, schematically X is represented by:



where

$$m_i = i + n_i \quad \text{iff} \quad X = S^0 \cup e^{m_1} \cup \cdots \cup e^{m_k}$$

and the attaching map for the jth cell

$$\varphi_j : S^{m_j - 1} \to S^0 \cup \cdots \cup e^{m_{j-1}}$$

when composed which the natural collapse

$$c_j : S^0 \cup \cdots \cup e^{m_{j-1}} \to S^{m_{j-1}}$$

coincides with

$$\theta_j : S^{m_j - 1} = S^{j-1+n_1+\cdots+n_j} \to S^{j-1+\cdots+n_{j-1}} = S^{m_{j-1}}$$

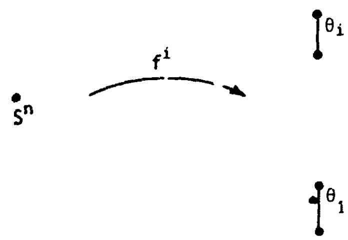
for $j = 1, \ldots, k$.

Example 2. Suppose given a complex

and a sequence of maps

$$f_i : S^n \to S^0 \cup e^{m_1} \cup \cdots \cup e^{m_i} \quad i = 0, \ldots, k$$

schematically



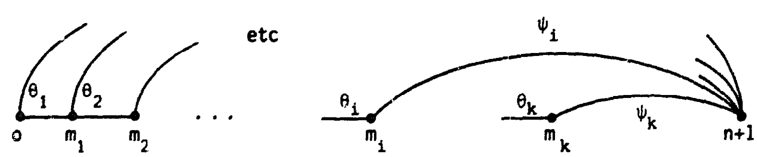
Let (remember we are in the stable category or range)

$$f = \bar{f}_0 + \cdots + \bar{f}_k : S^n \to X$$

where f_i is the composite

$$S^n \xrightarrow{f_i} S^0 \cup \cdots \cup e^{m_i} \subset X.$$

The complex $X_f e^{n+1}$ is then represented by the scheme

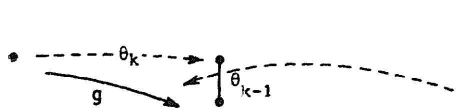


where ψ_i is the composition

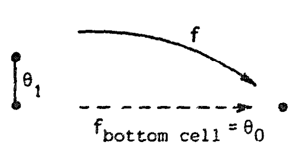
$$S^n \xrightarrow{f_i} S^0 \cup \cdots \cup e^{m_i} \longrightarrow S^{m_i}$$

and lies in $\pi^s_{n-m_i}$.

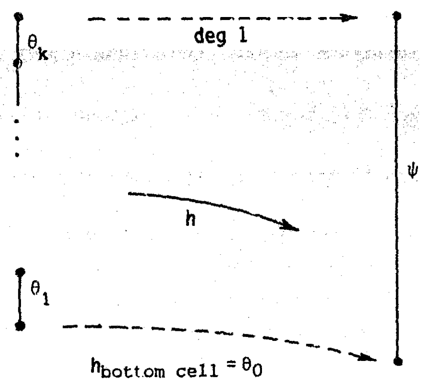
In terms of these schemes one can say that the Toda bracket $\langle \theta_0, \ldots, \theta_k \rangle$ is defined if there is a diagram of schemes



that is; the composition of g and the natural collapse onto the top cell is θ_k.



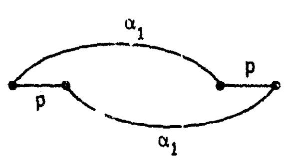
The composition $f \circ g$ is then a representative of the bracket. Put another way, a diagram



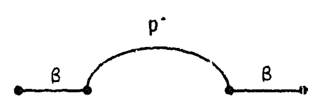
implies that $\langle \theta_0, \ldots, \theta_k \rangle$ is defined and contains the attaching map ψ of the right hand complex.

Note that a complex

implies $\langle \theta_0, \theta_1, \ldots, \theta_k \rangle$ is defined and contains θ. In terms of these diagrams, the complex $V(1)$ is represented by the diagrams



cell structure

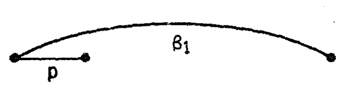


cohomology structure

The mapping cone of the map

$$S^0 \vee S^{2p(p-1)-2} \xrightarrow[p \vee \beta_1]{} S^0$$

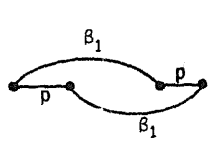
is represented by



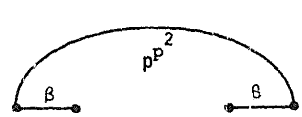
and the fact the composition

$$S^{2p(p-1)-2} \xrightarrow[\beta_1 \vee p]{} S^0 \vee S^{2p(p-1)-2} \xrightarrow[p \vee \beta_1]{} S^0$$

is null homotopic allows to construct a complex represented by the diagrams



cell structure



cohomology structure.

References

[1] H. Cartan and J.P. Serre. Cohomologie d'algebres d'Eilenberg–Mac Lane, Seminar ENS Paris 1954/55.

[2] G.E. Cooke, Replacing Homotopy Actions by Topological Actions Transaction, A.M.S. 237 (1978) 391–406.

[3] G.E. Cooke, Constructing Spaces with Interesting Z/p Cohomology via ε Actions on Loop Spaces (preprint).

[4] G.E. Cooke and L. Smith, Decompositions of co H-Spaces and Applications, Math. Zeit. (to appear).

[5] J. Harpur, Mod 3 Homotopy type of F_4, Localization in Group Theory and Homotopy Theory and Related Topics, Lecture Notes 418 (Springer-Verlag, Berlin) 58–67.

[6] P.J. Hilton, Homotopy Theory and Duality (Gordon and Breach, New York 1965).

[7] M. Mimura and H. Toda, On p Equivalences and p Universal Spaces, Comment. Math. Helv. 46 (1971) 87.

[8] S. Oka, The Stable Homotopy Groups of Spheres I, Hiroshima Math. Journal 1 (1971) 305–337.

[9] L. Smith, Homological Algebra and the Eilenberg–Moore Spectral Sequence, TAMS 129 (1967) 58–90.

[10] L. Smith, Cohomology of Stable Two Stage Postnikov Systems, Ill, J. of Math. 11 (1967) 310–329.

[11] L. Smith, Lectures on the Eilenberg–Moore Spectral Sequence, Springer Lecture Notes No. 134 (1970).

[12] L. Smith, Realizing Complex Bordism Modules, A.J. of Math. 92 (1970) 793–956.

[13] H. Toda, p-Primary Components of Homotopy Groups of Spheres IV, Memoirs Coll. Sci. Kyoto 32 (1959) 297–333.

[14] H. Toda, An Important Relation in the Homotopy Groups of Spheres, Proc. Jap. Acad. 43 (1967) 839–842.

[15] H. Toda, Algebra of Stable Homotopy of Z/p Spaces and Applications, J. Math. Kyoto Uni. 11 (1971) 197–251.

[16] Yamamoto, Algebra of Stable Homotopy of Moore Space, J. Math. Osaka City Uni. 14 (1963) 45–67.

[17] R. Zahler and E. Thomas, Nontriviality of the Stable Homotopy Element γ_1, J. Pure and Appl. Algebra 4 (1974) 189–203.